

A simple formula for gravitational MHV amplitudes

Andrew Hodges*

Mathematical Institute, University of Oxford, Oxford OX1 3LB, U.K.

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Abstract

A simple formula is given for the n -field tree-level MHV gravitational amplitude, based on soft limit factors. It expresses the full S_n symmetry naturally, as a determinant of elements of a symmetric $(n \times n)$ matrix.

1 Introduction

This note extends the material introduced in (Hodges 2011). That paper showed how BCFW recursion (Britto, Cachazo, Feng and Witten 2005) can be applied with $N=7$ super-symmetry to write down simplified expressions for all tree-level gravitational amplitudes. In particular, for MHV amplitudes this method yielded a recursion relation which avoids spurious double poles and gives a direct proof of the standard BGK expressions (Berends, Giele and Kuijf 1988), later justified on quite different grounds by Mason and Skinner (2009). Further sections of the paper developed a calculus of phase factors in which shorter and suggestive expressions for 6- and 7-field MHV amplitudes were given.

Now we pursue this program further by proving a new formula for the n -field MHV amplitude. This, the analogue of the Parke-Taylor formula in gauge theory, effects a great simplification. This new result also clarifies the momentum-twistor picture introduced in (Hodges 2011), by proving the existence of polynomials which express the content of the gravitational interaction.

*andrew.hodges@wadh.ox.ac.uk, <http://www.twistordiagrams.org.uk>.

2 The phase factor and soft factor definitions

In (Hodges 2011) the following useful definition of phase factors was given:

$$\begin{aligned}\psi_j^i &= \frac{[ij]}{\langle ij \rangle} \text{ (for } i \neq j) \\ \psi_i^i &= 0.\end{aligned}\tag{1}$$

We now make a different definition, using the symbol ϕ to avoid confusion with the ψ . For $i \neq j$ the definition is just the same, but when $i = j$ we make a significant change:

$$\begin{aligned}\phi_j^i &= \frac{[ij]}{\langle ij \rangle} \text{ (for } i \neq j) \\ \phi_i^i &= -\sum_{j \neq i} \frac{[ij]\langle jx \rangle \langle jy \rangle}{\langle ij \rangle \langle ix \rangle \langle iy \rangle}.\end{aligned}\tag{2}$$

This new quantity is the (negative of the) *universal gravitational soft factor* associated with adding the i th field to the others, as defined in (Nguyen, Spradlin, Volovich and Wen, 2009). Momentum conservation ensures that the definition is independent of the spinors x, y . Note that the spinorial weight of the ϕ_j^i is (-2) on each index, this remaining true for ϕ_i^i . It is most important to note that ϕ_i^i is only defined relative to a complete set of n momenta summing to zero; it has an implicit dependence on the the other $(n - 1)$ momenta.

This negative sign is chosen so that we have a convenient form for the vital linear relation, from which everything flows:

$$\sum_{j=1}^n \phi_j^i \pi_j^{A'} \pi_j^{B'} = 0.\tag{3}$$

It is also convenient to define

$$c_{ijk} = c^{ijk} = \{\langle ij \rangle \langle jk \rangle \langle ki \rangle\}^{-1},\tag{4}$$

so that the c_{ijk} are completely antisymmetric in their indices.

As in (Hodges 2011), we shall use square brackets round indices to indicate antisymmetrization without any $1/n!$ factor.

3 The new formula

Then the main result is that the reduced gravitational MHV amplitude \bar{M}_n is given simply by:

$$\bar{M}_n(12 \dots n) = (-1)^{n+1} \text{sgn}(\alpha\beta) c_{\alpha(1)\alpha(2)\alpha(3)} c^{\beta(1)\beta(2)\beta(3)} \phi_{[\alpha(4)}^{\beta(4)} \phi_{\alpha(5)}^{\beta(5)} \dots \phi_{\alpha(n)]}^{\beta(n)}, \quad (5)$$

where α and β are any permutations of $\{123 \dots n\}$.

The ± 1 factors for the signature of the permutations are obviously necessary. Otherwise, the overall sign is not of great importance, as the definition of the reduced amplitude is conventional. But the $(-1)^{n+1}$ ensures that $\bar{M}_3(123)$ is simply $c_{123}c^{123}$ and that for $n > 3$ the formula is consistent with the definition of \bar{M}_n given by the recursive relation in (Hodges 2011), as we shall soon show.

Equivalently, let Φ be the $n \times n$ symmetric matrix formed by the ϕ_j^i , and $|\Phi|_{ijk}^{rst}$ be the $(n-3) \times (n-3)$ minor determinant obtained by striking out rows i, j, k and columns r, s, t . Then

$$\bar{M}_n(12 \dots n) = (-1)^{n+1} \sigma(ijk, rst) c^{ijk} c_{rst} |\Phi|_{ijk}^{rst}, \quad (6)$$

where $\sigma(ijk, rst) = \text{sgn}((ijk12 \dots \cancel{j} \cancel{k} \dots n) \rightarrow (rst12 \dots \cancel{r} \cancel{s} \dots n))$.

We first establish that formulas (5) and (6) are well-defined, i.e. that they are independent of the permutations, and so enjoy S_n symmetry. We first show that

$$c_{123} |\Phi|_{rst}^{123} = -c_{124} |\Phi|_{rst}^{124} \quad (7)$$

Note that $\{r, s, t\}$ may overlap with $\{1, 2, 3, 4\}$, without restriction.

To do this, it is useful to define $f_j^i = \langle i1 \rangle \langle i2 \rangle \phi_j^i$. Then by (3),

$$\sum_{i=1}^n f_j^i = 0. \quad (8)$$

That is, the rows in the complete $n \times n$ matrix f_j^i all sum to zero. The identity to be shown is equivalent to:

$$|f|_{rst}^{123} = -|f|_{rst}^{124},$$

which is immediate from (8) and the elementary properties of determinants. But now similarly $c_{ijk} |\Phi|_{rst}^{ijk} = -c_{ijm} |\Phi|_{rst}^{ijm}$, and then any permutation can be composed from such transpositions. This completes the proof, and presents the S_n symmetry as a trivial consequence of (3).

The expression (5) is thus well-defined, completely symmetric, and also of the right spinorial weight. It remains to show that it satisfies the recursion relation as derived in (Hodges 2011), at equation (59):

$$\bar{M}_n(123 \dots n-1, n) = \sum_{p=3}^{n-1} \frac{[pn]}{\langle pn \rangle} \frac{\langle 1p \rangle \langle 2p \rangle}{\langle 1n \rangle \langle 2n \rangle} \bar{M}_{n-1}(\hat{1}_{(p)} 23 \dots \hat{p} \dots n-1), \quad (9)$$

where

$$\hat{1}_{(p)}] = \frac{(1+n)|p\rangle}{\langle 1p \rangle}, \quad \hat{1}_{(p)}\rangle = |1\rangle, \quad \hat{p}] = \frac{(p+n)|1\rangle}{\langle p1 \rangle}, \quad \hat{p}\rangle = |p\rangle, \quad (10)$$

so that $\hat{1}_{(p)} + \hat{p} = 1 + p + n$. The notation $\hat{1}_{(p)}$ is used to emphasise that the shifted momentum $\hat{1}$ is different in each of the $(n-3)$ terms, depending on p .

We can exploit the freedom of representation offered by the new formula to choose a helpful representation of the \bar{M}_{n-1} at each point of the recursion. The algebraic complexity arises mainly from the ‘shifted’ momenta $\hat{1}_{(p)}$ and \hat{p} , so we craftily put these within the set of three which do not appear in the determinant. In fact we choose the triple $\{12p\}$ for both the excluded rows and the excluded columns. We also note that

$$\frac{[pn]}{\langle pn \rangle} = \phi_p^n, \quad \frac{\langle 1p \rangle \langle 2p \rangle}{\langle 1n \rangle \langle 2n \rangle} c_{12p} = c_{12n},$$

so that the consistency of the recursion relation (9) is equivalent to showing:

$$\bar{M}_n = (-1)^n \sum_{p=3}^{n-1} \phi_p^n c_{12n} c^{12p} |\hat{\Phi}|_{12p}^{12p}. \quad (11)$$

Here the hatted $\hat{\Phi}$ is an $(n-1) \times (n-1)$ matrix in which the objects $\hat{\phi}_j^i$ are defined with respect to the $(n-1)$ shifted momenta $\{\hat{1}_{(p)}, 2, 3 \dots p-1, \hat{p}, p+1, \dots n-1\}$ summing to zero. Our choice of representation means that the only difference between $\hat{\phi}_j^i$ and ϕ_j^i is that within the p th term,

$$\hat{\phi}_k^k = \phi_k^k + \phi_n^k \frac{\langle 1n \rangle \langle pn \rangle}{\langle 1k \rangle \langle pk \rangle}. \quad (12)$$

Again it is useful to write:

$$f_j^i = \langle 1i \rangle \langle 2i \rangle \phi_j^i, \quad \hat{f}_j^i = \langle 1i \rangle \langle 2i \rangle \hat{\phi}_j^i, \quad \hat{f}_k^k = f_k^k + f_n^k \frac{\langle pn \rangle \langle 2k \rangle}{\langle 2n \rangle \langle pk \rangle}. \quad (13)$$

We define F as the $(n-3) \times (n-3)$ matrix with entries f_j^i for $3 \leq i, j \leq n-1$, with its minor determinants indicated in the same way as for Φ , and \hat{F} analogously. So in these terms, it is required to prove that:

$$(-1)^n \prod_{k=3}^{n-1} \langle 1k \rangle \langle 2k \rangle \bar{M}_n = c_{12n} c^{12n} \sum_{p=3}^{n-1} f_p^n |\hat{F}|_p^p. \quad (14)$$

To do this, we expand the $f_p^n |\hat{F}|_p^p$, ordering the sum by the number of shift-correction factors used in each term of the expansion. The zeroth order term has no correction terms, and gives $f_p^n |F|_p^p$. A typical first order term comes from taking just one correction, say for f_q^q in one of the terms of the summation, say the p th, where $p \neq q$. This contributes

$$f_p^n f_q^n |F|_{pq}^{pq} \frac{\langle pn \rangle \langle 2q \rangle}{\langle 2n \rangle \langle pq \rangle}.$$

Summing over all p and all q , and so symmetrising over p and q , the Schouten identity gives this simple expression for the sum of all first-order corrections:

$$\sum_{3 \leq p < q \leq n-1} f_p^n f_q^n |F|_{pq}^{pq}.$$

A typical second order term comes from taking two corrections in the p th term of the summation, say from f_q^q and f_r^r , thus contributing

$$f_p^n f_q^n f_r^n |F|_{pqr}^{pqr} \frac{\langle pn \rangle \langle 2q \rangle}{\langle 2n \rangle \langle pq \rangle} \frac{\langle pn \rangle \langle 2r \rangle}{\langle 2n \rangle \langle pr \rangle}.$$

Adding in the contribution from f_p^p and f_r^r in the q th term of the summation, and f_p^p and f_q^q in the r th term of the summation, it is clear that every disjoint set $\{p, q, r\}$ contributes

$$f_p^n f_q^n f_r^n |F|_{pqr}^{pqr} \frac{\langle pn \rangle \langle 2q \rangle}{\langle 2n \rangle \langle pq \rangle} \frac{\langle pn \rangle \langle 2r \rangle}{\langle 2n \rangle \langle pr \rangle} + (p \leftrightarrow q) + (p \leftrightarrow r),$$

which is readily seen to be

$$f_p^n f_q^n f_r^n |F|_{pqr}^{pqr},$$

and so contributing a total

$$\sum_{3 \leq p < q < r \leq n-1} f_p^n f_q^n f_r^n |F|_{pqr}^{pqr}.$$

The same simplification occurs at every order, so it remains only to show that:

$$(-1)^n \prod_{k=3}^{n-1} \langle 1k \rangle \langle 2k \rangle \bar{M}_n = c_{12n} c^{12n} \sum_{i=1}^{n-3} \sum_{3 \leq p_1 < p_2 \dots p_i \leq n-1} f_{p_1}^n \dots f_{p_i}^n |F|_{p_1 p_2 \dots p_i}^{p_1 p_2 \dots p_i} . \quad (15)$$

But this too is simple. Consider the $(n-3) \times (n-3)$ matrix H with entries

$$h_j^i = f_j^i + \delta_j^i f_j^n \text{ for } 3 \leq i, j \leq n-1 .$$

Thus H has the same entries as F , but with the addition of f_i^n to elements down the main diagonal. Each row of H sums to zero, by (8), so its determinant $|H|$ vanishes. On the other hand, we can also expand $|H|$ in terms of the number of f_i^n factors. The zeroth order part is just $|F|$. The i th order part is

$$\sum_{3 \leq p_1 < p_2 \dots p_i \leq n-1} f_{p_1}^n \dots f_{p_i}^n |F|_{p_1 p_2 \dots p_i}^{p_1 p_2 \dots p_i} .$$

It follows that

$$0 = |H| = |F| + \sum_{i=1}^{n-3} \sum_{3 \leq p_1 < p_2 \dots p_i \leq n-1} f_{p_1}^n \dots f_{p_i}^n |F|_{p_1 p_2 \dots p_i}^{p_1 p_2 \dots p_i} .$$

Thus the claim we are checking reduces to

$$(-1)^n \prod_{k=3}^{n-1} \langle 1k \rangle \langle 2k \rangle \bar{M}_n = -c_{12n} c^{12n} |F| ,$$

which indeed is true, being equivalent to

$$\bar{M}_n = (-1)^{n+1} c_{12n} c^{12n} |\Phi|_{12n}^{12n} .$$

This observation concludes the recursive proof of the new formula for \bar{M}_n .

It is striking that whilst determinants are naturally thought of as generating *anti-symmetry*, the minor determinants of the symmetric Φ matrix naturally yield a *S_n-symmetry* — exactly as needed for a gravitational amplitude. This suggests scope for generalization beyond MHV tree amplitudes.

The new formula is much simpler than the BGK-Mason-Skiner expression, in a very concrete sense. Given numerical data for the spinors, it requires only $O(n^2)$ operations to find the ϕ_j^i , after which the determinant of a symmetric matrix of order $(n-3)$ must be computed. This is easily achieved in (better than) $O(n^3)$ time. In contrast, the BGK-Mason-Skiner formula requires summation

over $(n - 3)!$ terms and so grows exponentially. The new gravitational formula even compares well with gauge theory, where the simplicity of the Parke-Taylor formula emerges only after the separation into $(n - 1)!/2$ colour-order sectors, all of which must be considered. For a general gluon interaction, therefore, the complexity is exponential in n . This formula might be seen as an indication of the emergent simplicity of gravitational scattering, notably advanced by Nima Arkani-Hamed, Freddy Cachazo and Jared Kaplan (2008). It can hardly be doubted that further enormous simplifications can be achieved.

4 Illustrative examples

The new formula includes all the expressions given in (Hodges 2011) in terms of the ψ_j^i and extends them by changing to the ϕ_j^i . Thus we have

$$\begin{aligned}\bar{M}_3(123) &= \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \\ \bar{M}_4(1234) &= \frac{\phi_4^1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 23 \rangle \langle 34 \rangle \langle 42 \rangle}, \\ \bar{M}_5(12345) &= \frac{\phi_{[4}^1 \phi_5^2]}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle},\end{aligned}\tag{16}$$

but now we also have, for instance,

$$\begin{aligned}\bar{M}_4(1234) &= -\frac{\phi_4^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \\ \bar{M}_5(12345) &= \frac{\phi_{[4}^4 \phi_5^5]}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}.\end{aligned}\tag{17}$$

Likewise for $n = 6$ we have the new expression found in (Hodges 2011):

$$\bar{M}_6(123456) = \frac{\phi_{[4}^1 \phi_5^2 \phi_6^3]}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 45 \rangle \langle 56 \rangle \langle 64 \rangle},\tag{18}$$

but also:

$$\begin{aligned}\bar{M}_6(123456) &= -\frac{\phi_{[4}^4 \phi_5^5 \phi_6^6]}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\ &= \frac{\phi_{[4}^4 \phi_5^5 \phi_6^3]}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \\ &= -\frac{\phi_{[4}^4 \phi_5^2 \phi_6^3]}{\langle 15 \rangle \langle 56 \rangle \langle 61 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}.\end{aligned}\tag{19}$$

For $n = 7$, expression (77) in (Hodges 2011) was offered as the shortest identifiable formula:

$$\begin{aligned}
\bar{M}_7(1234567) &= \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 \psi_6^7}{\langle 12 \rangle \langle 27 \rangle \langle 71 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} + \frac{\psi_{[4}^3 \psi_5^1 \psi_6^2 \psi_3^7}{\langle 12 \rangle \langle 27 \rangle \langle 71 \rangle \langle 64 \rangle \langle 45 \rangle \langle 56 \rangle} \\
&+ \frac{\psi_{[3}^4 \psi_5^1 \psi_6^2 \psi_4^7}{\langle 12 \rangle \langle 27 \rangle \langle 71 \rangle \langle 36 \rangle \langle 65 \rangle \langle 53 \rangle} + \frac{\psi_{[3}^5 \psi_4^1 \psi_6^2 \psi_5^7}{\langle 12 \rangle \langle 27 \rangle \langle 71 \rangle \langle 34 \rangle \langle 46 \rangle \langle 63 \rangle} \\
&+ \frac{\psi_{[3}^7 \psi_4^1 \psi_5^2 \psi_7^6}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle}. \tag{20}
\end{aligned}$$

By making copious use of the 6-point identity noted in equation (65) of (Hodges 2011), this can be rewritten with a common denominator:

$$\begin{aligned}
\bar{M}_7(1234567) &= \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 \psi_6^7}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \frac{\langle 16 \rangle \langle 26 \rangle}{\langle 17 \rangle \langle 27 \rangle} \\
&+ \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 \psi_3^7}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \frac{\langle 13 \rangle \langle 23 \rangle}{\langle 17 \rangle \langle 27 \rangle} \\
&+ \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 \psi_4^7}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \frac{\langle 14 \rangle \langle 24 \rangle}{\langle 17 \rangle \langle 27 \rangle} \\
&+ \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 \psi_5^7}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \frac{\langle 15 \rangle \langle 25 \rangle}{\langle 17 \rangle \langle 27 \rangle} \\
&+ \frac{\psi_{[3}^7 \psi_4^1 \psi_5^2 \psi_7^6}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle}. \tag{21}
\end{aligned}$$

But the first four terms are now naturally gathered together into the universal soft factor ϕ_7^7 , so giving

$$\begin{aligned}
\bar{M}_7(1234567) &= \frac{\psi_{[3}^6 \psi_4^1 \psi_5^2 (-\phi_7^7)}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} + \frac{\psi_{[3}^7 \psi_4^1 \psi_5^2 \psi_7^6}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle} \\
&= -\frac{\phi_{[3}^1 \phi_4^2 \phi_5^6 \phi_7^7}{\langle 12 \rangle \langle 26 \rangle \langle 61 \rangle \langle 34 \rangle \langle 45 \rangle \langle 53 \rangle}. \tag{22}
\end{aligned}$$

in agreement with the new general formula. It was actually this observation that suggested the definition of ϕ_i^i , and hence the extension to $n > 7$.

5 The momentum-twistor numerator

It was conjectured in (Hodges 2011), at equation (97), that a natural polynomial arises when we express the n -point gravitational MHV amplitude in terms of the momentum-twistor space introduced in (Hodges 2009). That is, we define $N_n(12 \dots n)$ by

$$\tilde{M}_n(123 \dots n) = \frac{N_n(123 \dots n)}{\prod_i \langle i, i+1 \rangle \prod_{i < j} \langle ij \rangle}, \quad (23)$$

and the conjecture is that N_n is a polynomial (rather than a rational function).

The proof of this conjecture now follows immediately from cunning choice of the representation

$$\bar{M}_n(12 \dots n) = (-1)^n \frac{\phi_2^1 \phi_4^3 \psi_6^5 \phi_7^7 \phi_8^8 \phi_9^9 \dots \phi_n^n}{\langle 13 \rangle \langle 35 \rangle \langle 51 \rangle \langle 24 \rangle \langle 46 \rangle \langle 62 \rangle}. \quad (24)$$

Translating this into momentum twistors as defined by the ordering $(123 \dots n)$, it is obvious that just two denominator factors of $\langle 12 \rangle$ will occur in each term of the expansion. These are safely absorbed in $\prod_i \langle i, i+1 \rangle \prod_{i < j} \langle ij \rangle$, and the numerator function is therefore not singular in $\langle 12 \rangle$. By cyclicity, it cannot be singular in any other $\langle i, i+1 \rangle$. A similar argument applies to all the other $\langle ij \rangle$ factors, and so it must be a polynomial.

For $n = 5$ it was shown in (Hodges 2011) that the polynomial is the area of a pentagon defined by the 5 points in \mathbb{C}^2 defined by $\langle 1234 \rangle \pi_5^{A'}$ etc. The general polynomial is of degree $(n-3)$ in the twistors and degree $(n-3)(n-4)/2$ in I . So it is a $(n-3)$ -degree polynomial in the $n!/(n-4)!4!$ objects like $\langle 1234 \rangle \pi_5^{A'} \dots \pi_n^{N'}$. It remains to be seen how this can be characterized geometrically.

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7 References

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